Worcester County Mathematics League

Varsity Meet 2
November 12, 2014

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 – November 12, 2014
Round 1: Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Express the following as a single, simplified fraction:

$$\frac{\frac{1}{2}+\frac{1}{3}+0.4}{\frac{3}{4}+\frac{1}{5}+0.375}$$

2. Jill bought a car, then sold it to Kim, making a 10% profit. Kim spent \$450 on improvements, then sold the car to Larry for 10% above her total expenses. Larry sold the car to Matthew for \$1617, taking a loss of 30%. How much did Jill pay for the car?

3. Find the value of the following infinite continued fraction, where the values 3 and 6 continue to alternate after the first 3:

$$3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\ddots}}}}$$

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________ dollars

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 – November 12, 2014
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Find the value of t, if

$$2 - 5[3t + 2(6 - t)] = -7(t - 4)$$

2. If the graphs of $2x - 3y = 1$ and $5x - ay = b$ do not intersect, and the graph of $5x - ay = b$ contains the point $(1, 1)$, determine the value of $\frac{b}{a}$.

3. Every year for her birthday, Kathy's three uncles give her a fixed dollar amount as a gift. Kathy invests each gift in a different company. On her fifteenth birthday, her gift from Uncle Andy made a profit of 8%, her gift from Uncle Bob lost 7%, and her gift from Uncle Carl lost 2%, for a net gain of \$1.65. On her sixteenth birthday, her gift from Uncle Andy made a profit of 11%, her gift from Uncle Bob lost 2%, and her gift from Uncle Carl made a profit of 10%, for a net gain of \$6.60. The sum of the three yearly gifts is \$80. How much is Uncle Carl's gift to Kathy each year?

ANSWERS

(1 pt.) 1. $t =$ ____________________

(2 pts.) 2. ____________________

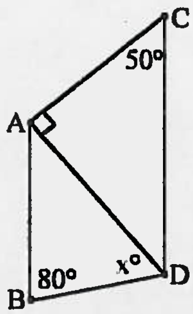
(3 pts.) 3. ____________________ dollars

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 – November 12, 2014
Round 3: Parallel Lines and Polygons

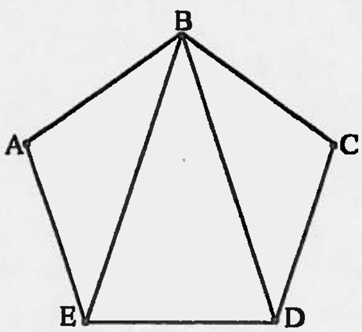
All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1.



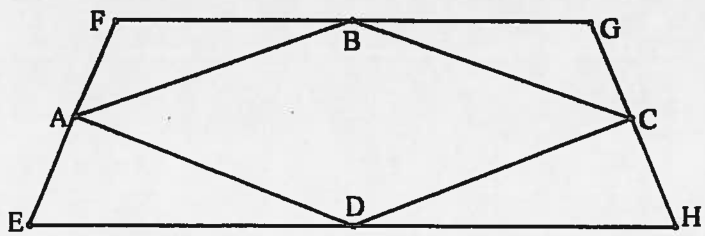
In the figure, $\overline{AB} \parallel \overline{CD}$.
Find the value of x.

2.



The pentagon $ABCDE$ is a regular pentagon.
Find the measure of $\angle EBD$.

3. Quadrilateral $EFGH$ is an isosceles trapezoid, with $\overline{FG} \parallel \overline{EH}$ and $EF = GH$. $FG = 10$ and $EH = 14$. The trapezoid's altitude has length 5, and A, B, C, and D are the midpoints of $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, and $\overline{EH}$, respectively.
Find $AB + BC + CD + DA$.



ANSWERS

(1 pt.) 1. $x =$ ____________________

(2 pts.) 2. ____________________ degrees

(3 pts.) 3. ____________________ units

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 – November 12, 2014
Round 4: Sequences and Series

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Find the 32^{nd} term of the arithmetic sequence

$$-11, -7, -3, \ldots$$

2. The first term of a geometric progression is 375 and the fourth term is 192. Find the sum of the first three terms.

3. The numbers a, b, and c form an arithmetic progression, and their sum is 15. If 1, 3, and 9 are added to a, b, and c respectively, the resulting three numbers form a geometric progression. Find all possible ordered triples (a, b, c).

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 2 – November 12, 2014
Round 5: Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Find the matrix X that satisfies

$$2X + 2\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = 6\begin{bmatrix} \frac{1}{3} & 2 \\ -1 & \frac{1}{2} \end{bmatrix}$$

2. Find the value(s) of x which satisfy

$$\begin{vmatrix} x & 9 & 6 \\ -5 & 1 & 4 \\ 3 & x & -2 \end{vmatrix} = -260$$

3. Find all ordered pairs (a, b) for which $A = \begin{bmatrix} 6a & 2b \\ 5a & 3b \end{bmatrix}$ is an invertible matrix such that $A = A^{-1}$.

ANSWERS

(1 pt.) 1. $X = \begin{bmatrix} & \\ & \end{bmatrix}$

(2 pts.) 2. $x =$ ____________________

(3 pts.) 3. ________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

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Team Round

All answers must be in simplest exact form, and be written on the separate team answer sheet

1. The sequence below is formed as follows: the first three terms are three consecutive odd numbers starting with 1; the next three terms are three consecutive even numbers starting with 2; the next three terms are three consecutive odd numbers starting with 3; and so on. Find the sum of the first one hundred terms.

$$1, 3, 5, 2, 4, 6, 3, 5, 7, 4, 6, 8, \ldots$$

2. If $A = \begin{bmatrix} 7 & 2 & -8 \\ 3 & 5 & 1 \\ -6 & 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 1 & -9 \\ 2 & 6 & 0 \\ -7 & -1 & -3 \end{bmatrix}$, find

$$A^2 - AB - BA + B^2$$

3. One regular heptagon has a side length of 7, and a second regular heptagon has a side length of 24. A third regular heptagon has an area which equals the sum of the areas of the first two heptagons. Find the length of a side of the third heptagon.

4. Find the area of the region in the coordinate plane containing all points (x, y) that are in the first quadrant, lie between the line $y = 2x - 1$ and the line through the point $(2, 6)$ that is parallel to $y = 2x - 1$, and satisfy $x \leq 4$.

5. The infinite series below is formed by alternately multiplying the previous term by $-\frac{1}{3}$ and $-\frac{1}{5}$. Find its sum.

$$\frac{1}{3} - \frac{1}{15} + \frac{1}{45} - \frac{1}{225} + \cdots$$

6. Find all real number solutions to:

$$7^{2x+4} = \left(\frac{1}{49}\right)^{9-2x}$$

7. Evaluate $12\left(\sin\frac{\pi}{12}\right) \cdot \left(\cos\frac{\pi}{12}\right)$.

8. How many three-digit positive integers have their digits in strictly increasing order (as in 247, for example)?

9. Evaluate $105_6 + 342_5$ and express the result in base 7.

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Answers

Round 1: Fractions, Decimals, and Percents

1. $\frac{148}{159}$

2. 1500

3. $\sqrt{11}$

Tantasqua, Assabet Valley, Algonquin

Round 2: Algebra 1

1. 43

2. $-\frac{1}{3}$

3. 25

Southbridge, Quaboag, QSC

Round 3: Parallel Lines and Polygons

1. 60

2. 36

3. 26

Millbury, QSC, Quaboag

Round 4: Sequences and Series

1. 113

2. 915

3. $(15, 5, -5)$ and $(3, 5, 7)$

Bartlett, Clinton, Hudson

Round 5: Matrices and Systems of Equations

1. $\begin{bmatrix} -2 & 4 \\ -7 & \frac{1}{2} \end{bmatrix}$

2. 5 and -13

3. $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Bancroft, Hudson, QSC

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ANSWERS: Team Round

1. 1915

2. $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$

3. 25

4. $\frac{47}{4}$ or $11\frac{3}{4}$ or 11.75

5. $\frac{2}{7}$

6. 11

7. 3

8. 84

9. 255

Shrewsbury, QSC, Doherty, Holliston, QSC, Doherty, Leicester, Mass Academy, Shrewsbury

WORCESTER COUNTY MATHEMATICS LEAGUE

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Team Round Answer Sheet

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM, AND BE WRITTEN ON THIS TEAM ANSWER SHEET.

(2 points each)

1. ______________________

2. [ ]

3. ____________________ units

4. ________________ sq. units

5. ______________________

6. ______________________

7. ______________________

8. ______________________

9. ______________________7

WORCESTER COUNTY MATHEMATICS LEAGUE

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Solutions

Round 1: Fractions, Decimals, and Percents

1.

$$\frac{\frac{1}{2}+\frac{1}{3}+0.4}{\frac{3}{4}+\frac{1}{5}+0.375}=\frac{\frac{1}{2}+\frac{1}{3}+\frac{4}{10}}{\frac{3}{4}+\frac{1}{5}+\frac{375}{1000}}=\frac{\frac{1}{2}+\frac{1}{3}+\frac{2}{5}}{\frac{3}{4}+\frac{1}{5}+\frac{3}{8}}=\frac{\left(\frac{15+10+12}{30}\right)}{\left(\frac{30+8+15}{40}\right)}=\frac{\left(\frac{37}{30}\right)}{\left(\frac{53}{40}\right)}=\frac{37\times 40}{53\times 30}$$
$$=\frac{37\times 4}{53\times 3}=\mathbf{\frac{148}{159}}$$

2. <u>Solution 1</u>: Since \$1617 is 30% less than what Larry paid for the car (call this value x), $x\cdot\frac{70}{100}=1617$, so $x=1617\div\frac{70}{100}=2130$. Then \$2130 is 10% more than Kim's total investment (call this value y), i.e. $2310=\frac{110}{100}\cdot y$, giving $y=2310\cdot\frac{100}{110}=2100$. This means Jill sold the car for $\$2100-\$450=\$1650$. This amount is 10% more than Jill paid for the car (call this value z), so $z\cdot\frac{110}{100}=1650\implies z=1650\cdot\frac{100}{110}=\mathbf{1500}$.

<u>Solution 2</u>: Let x be the amount Jill paid for the car. Then she sold the car to Kim for $1.1x$, who added \$450 in improvements, and sold the car to Larry for $1.1(1.1x+450)$. Finally, Larry sold the car for $0.7\times 1.1(1.1x+450)=1617$. Solving, we have:

$$1.1x+450=1617\div(0.7\times 1.1)=1617\div 0.77=2100$$
$$\implies 1.1x=2100-450=1650\implies x=1650\div 1.1=\mathbf{1500}$$

3. Let $x=3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\ddots}}}}$. Then $x-3=\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\ddots}}}}=\cfrac{1}{3+\cfrac{1}{6+(x-3)}}$.

Then we have: $x-3=\cfrac{1}{3+\cfrac{1}{6+(x-3)}}=\cfrac{1}{3+\cfrac{1}{x+3}}=\cfrac{1}{\frac{3x+9}{x+3}+\frac{1}{x+3}}=\cfrac{1}{\frac{3x+10}{x+3}}=\frac{x+3}{3x+10}$

$\implies(x-3)(3x+10)=x+3\implies 3x^2+x-30=x+3\implies 3x^2=33\implies x^2=11$.
Then $x=\pm\sqrt{11}$. However, in the given problem x must be positive, so $x=\mathbf{\sqrt{11}}$.

Round 2: Algebra 1

1.

$$2-5[3t+2(6-t)]=-7(t-4)$$
$$\implies 2-5[3t+12-2t]=-7t+28$$
$$\implies 2-15t-60+10t=-7t+28$$
$$\implies -58-5t=-7t+28\implies 2t=86\implies t=43$$

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2. Solution 1: For the two equations' graphs not to intersect, the two equations must represent parallel lines, and therefore have proportional coefficients:

$$\frac{2}{5} = \frac{-3}{-a} \Rightarrow -2a = -15 \Rightarrow a = \frac{15}{2}$$

Since $(1, 1)$ is a solution to $5x - ay = b$, $5 \times 1 - \frac{15}{2} \times 1 = b \Rightarrow 5 - \frac{15}{2} = b \Rightarrow b = -\frac{5}{2}$

(Note that for these values of a and b, the two given lines do not coincide, so they do indeed have no solutions in common.)

Then $b \div a = -\frac{5}{2} \div \frac{15}{2} = -\frac{5}{2} \times \frac{2}{15} = -\frac{1}{3}$

Solution 2: If the two lines do not intersect, they must have the same slope. We find the slope of the lines by putting them in slope-intercept form:

$2x - 3y = 1 \quad \Rightarrow \quad 3y = 2x - 1 \quad \Rightarrow \quad y = \frac{2}{3}x - \frac{1}{3}$ so the slope is $\frac{2}{3}$.

$5x - ay = b \quad \Rightarrow \quad ay = 5x - b \quad \Rightarrow \quad y = \frac{5}{a}x - \frac{b}{a}$ so the slope is $\frac{5}{a}$.

Then $\frac{2}{3} = \frac{5}{a}$, and the second line can be written $y = \frac{2}{3}x - \frac{b}{a}$. Plugging in the coordinates from $(1, 1)$, we have $1 = \frac{2}{3} \times 1 - \frac{b}{a}$. Solving for $\frac{b}{a}$ gives $\frac{b}{a} = \frac{2}{3} - 1 = -\frac{1}{3}$

3. If b represents the amount Uncle Bob gives each year and c represents the amount Uncle Carl gives, then Uncle Andy gives $80 - b - c$. For Kathy's fifteenth birthday, we have:

$$0.08(80 - b - c) - 0.07b - 0.02c = 1.65$$
$$\Rightarrow 6.40 - 0.08b - 0.08c - 0.07b - 0.02c = 1.65$$
$$\Rightarrow -0.15b - 0.1c = -4.75$$
$$\Rightarrow 15b + 10c = 475$$
$$\Rightarrow 3b + 2c = 95$$

For Kathy's sixteenth birthday, we have

$$0.11(80 - b - c) - 0.02b + 0.10c = 6.60$$
$$\Rightarrow 8.80 - 0.11b - 0.11c - 0.02b + 0.1c = 6.60$$
$$\Rightarrow -0.13b - 0.01c = -2.20$$
$$\Rightarrow 13b + c = 220$$
$$26b + 2c = 440$$

Using elimination (subtracting the first equation from the second), we have $23b = 345$, so $b = \frac{345}{23} = 15$. Then $13b + c = 220$ implies

$$c = 220 - 13b = 220 - 13 \times 15 = 220 - 195 = 25$$

Round 3: Parallel Lines and Polygons

1. The sum of the angles in ΔACD is 180°, so $m\angle ADC = 180° - 90° - 50° = 40°$. Then $m\angle BDC = x° + 40°$. Since $\angle BDC$ and $\angle DBA$ are same-side interior angles formed by parallel line segments, they are supplementary. So $80° + x° + 40° = 180°$, and $\boldsymbol{x = 60}$.

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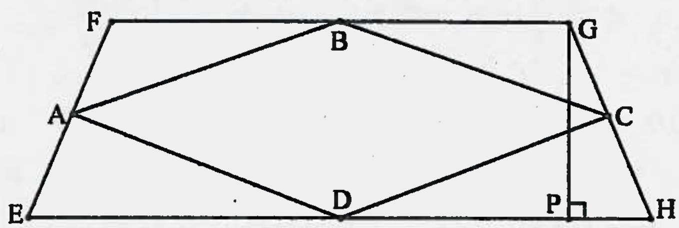
2. The measure of each angle in a regular pentagon is $\frac{180° \times (5-2)}{5} = 108°$. ΔABE is isosceles, so its two base angles, $\angle AEB$ and $\angle AEB$ are congruent, meaning they must each have measure $\frac{180°-108°}{2} = 36°$. Applying the same reasoning to ΔBDC, $m\angle CBD = m\angle CDB = 36°$. Since $m\angle ABC = 108°$, $m\angle EBD = 108° - 36° - 36° = \mathbf{36°}$.

3. Solution 1: The length of $\overline{GE}$ can be found using the Pythagorean Theorem, after first noting that, since $EFGH$ is isosceles, the distance EP in the figure below is $14 - \frac{(14-10)}{2} = 12$. Then $GE = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$. As C and D are the respective midpoints of $\overline{GH}$ and $\overline{EH}$, ΔEHG is similar to ΔDHC, with a scale factor of 2. Then $CD = \frac{1}{2}GE = 6.5$.



Furthermore, ΔEGF is similar to ΔABF with a scale factor of 2, indicating that $AB = 6.5$ also. This reasoning can be repeated using $\overline{FH}$ instead of $\overline{EG}$ to find $AD = BC = 6.5$. Then $AB + BC + CD + DA = 4 \times 6.5 = \mathbf{26}$.

Solution 2: Since A and C are the respective midpoints of $\overline{FE}$ and $\overline{GH}$, the length of $\overline{AC}$ is the mean of the lengths of $\overline{FG}$ and $\overline{EH}$. That is, $AC = \frac{10+14}{2} = 12$. $\overline{AC}$ is parallel to the two bases, and therefore perpendicular to the altitude $\overline{BD}$. So $\overline{AC}$ and $\overline{BD}$ divide rhombus $ABCD$ into four right triangles, each of which has one leg of length $\frac{12}{2}$ and one of length $\frac{5}{2}$. This means each hypotenuse has length $\sqrt{\left(\frac{12}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{169}{4}} = \frac{13}{2} = 6.5$. Then $AB + BC + CD + DA = 6.5 + 6.5 + 6.5 + 6.5 = \mathbf{26}$.

Round 4: Sequences and Series

1. The common difference between terms is $(-7) - (-11) = 4$. So the 32nd term is

$$-11 + (4 \times 31) = \mathbf{113}$$

2. The common ratio r must satisfy $375 \times r^3 = 192$. Solving for r, we have

$$r = \sqrt[3]{\frac{192}{375}} = \sqrt[3]{\frac{64}{125}} = \frac{\sqrt[3]{64}}{\sqrt[3]{125}} = \frac{4}{5}$$

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Then (if a_n represents the n^{th} term) $a_1 = 375$, $a_2 = 375 \times \frac{4}{5} = 300$, and $a_3 = 300 \times \frac{4}{5} = 240$, so that

$$a_1 + a_2 + a_3 = 375 + 300 + 240 = 915$$

3. Solution 1: Since the numbers form an arithmetic progression, $b - a = c - b$, i.e. $2b = a + c$. Adding b to both sides of this equation, we have $3b = a + b + c$, which we are told equals 15. Then $b = \frac{15}{3} = 5$.

Next, $\frac{a+1}{b+3} = \frac{b+3}{c+9}$, which yields $(a+1)(c+9) = (b+3)^2$. We can substitute in $b = 5$ and $c = 10 - a$ (from isolating c in the above equations), to get

$$(a+1)(10 - a + 9) = (5+3)^2$$
$$(a+1)(19 - a) = 8^2$$
$$19a - a^2 + 19 - a = 64$$
$$a^2 - 18a - 45 = 0$$
$$(a - 15)(a - 3) = 0$$

This means $a = 15$ or $a = 3$. When $a = 15$, $c = 10 - 15 = -5$, and when $a = 3$, $c = 10 - a = 7$. Hence the two solutions are

$$\mathbf{(3, 5, 7)} \text{ and } \mathbf{(15, 5, -5)}.$$

Solution 2: Let x be the common difference in the arithmetic progression. Then $a = b - x$ and $c = b + x$, and $a + b + c = 15$ yields $(b - x) + b + (b + x) = 15$, i.e. $3b = 15$, so $b = 5$.

Using these values for a, b, and c in the geometric progression, we have:

$$a + 1 = (b - x) + 1 = (5 - x) + 1 = 6 - x$$
$$b + 3 = 5 + 3 = 8$$
$$c + 9 = (b + x) + 9 = (5 + x) + 9 = 14 + x$$

which gives $\frac{6-x}{8} = \frac{8}{14+x} \implies (14+x)(6-x) = 64 \implies -x^2 - 8x + 84 = 64$
$\implies x^2 + 8x - 20 = 0 \implies (x+10)(x-2) = 0 \implies x = -10, 2$

When $x = -10$, $(a,b,c) = \big(5 - (-10), 5, 5 + (-10)\big) = \mathbf{(15, 5, -5)}$ and when $x = 2$, $(a,b,c) = (5 - 2, 5, 5 + 2) = \mathbf{(3, 5, 7)}$.

Round 5: Matrices and Systems of Equations

1. $2X + 2\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = 6\begin{bmatrix} \frac{1}{3} & 2 \\ -1 & \frac{1}{2} \end{bmatrix} \implies 2X + \begin{bmatrix} 6 & 4 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ -6 & 3 \end{bmatrix} \implies 2X = \begin{bmatrix} -4 & 8 \\ -14 & 1 \end{bmatrix}$

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$$\Rightarrow \quad X = \frac{1}{2}\begin{bmatrix} -4 & 8 \\ -14 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -7 & \frac{1}{2} \end{bmatrix}$$

2. To find the determinant of the 3×3 matrix, copy the first two columns after the third column, and multiply diagonally as indicated:

$$\begin{bmatrix} x & 9 & 6 \\ -5 & 1 & 4 \\ 3 & x & -2 \end{bmatrix} \begin{matrix} x & 9 \\ -5 & 1 \\ 3 & x \end{matrix}$$

The products of three quantities on the lines with negative slopes are subtracted from the sum of the products of the quantities on the lines with the positive slopes to get

$$-2x + 108 - 30x - 18 - 4x^2 - 90 = -260$$
$$-4x^2 - 32x + 260 = 0$$
$$x^2 + 8x - 65 = 0$$
$$(x+13)(x-5) = 0 \qquad \text{Then } x = -13, 5$$

3. $A = A^{-1}$ yields $\begin{bmatrix} 6a & 2b \\ 5a & 3b \end{bmatrix} = \frac{1}{18ab - 10ab}\begin{bmatrix} 3b & -2b \\ -5a & 6a \end{bmatrix} = \frac{1}{8ab}\begin{bmatrix} 3b & -2b \\ -5a & 6a \end{bmatrix}$, where $8ab \neq 0$ (or A would not be invertible).

This gives a system of four equations:

$$6a = \frac{1}{8ab}3b \quad \Rightarrow \quad 48a^2b = 3b \quad \Rightarrow \quad a^2 = \frac{1}{16} \quad \Rightarrow \quad a = \pm\frac{1}{4}$$
$$2b = \frac{1}{8ab}(-2b) \quad \Rightarrow \quad 8ab = -1$$
$$5a = \frac{1}{8ab}(-5a) \quad \Rightarrow \quad 8ab = -1$$
$$3b = \frac{1}{8ab}6a \quad \Rightarrow \quad 24ab^2 = 6a \quad \Rightarrow \quad b^2 = \frac{1}{4} \quad \Rightarrow \quad b = \pm\frac{1}{2}$$

(We are able to divide the equations by a and b above since neither variable can equal zero, as we know that $8ab \neq 0$.)

We check the four possible ordered pairs $\left(\frac{1}{4}, \frac{1}{2}\right)$, $\left(-\frac{1}{4}, \frac{1}{2}\right)$, $\left(\frac{1}{4}, -\frac{1}{2}\right)$, $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ in the equation $8ab = -1$ to see that $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and $\left(\frac{1}{4}, -\frac{1}{2}\right)$ are the two solutions.

Team Round

1. Solution 1: We can sum each triple of consecutive odd or even numbers separately first. To do this, note that the sum of three numbers in an arithmetic progression is 3

times the middle number, since $(x-d)+x+(x+d)=3x$. So, for example, $2+4+6=3\times 4$. Then the sum of the first 99 terms is

$$3\times 3+3\times 4+\cdots+3\times 35=3\sum_{i=3}^{35} i=3\times\left(\sum_{i=1}^{35} i-\sum_{i=1}^{2} i\right)=3\times\left(\frac{35\times 36}{2}-\frac{2\times 3}{2}\right)$$

$$=3\times(627)=1881$$

Finally, the one hundredth term is 34, giving a total of $1881+34=\mathbf{1915}$.

Solution 2: The given sequence comprises three arithmetic subsequences as follows:

The first subsequence contains $a_1, a_4, a_7, \ldots, a_{100}$, which are 1, 2, 3, …, 34.
The second subsequence contains $a_2, a_5, a_8, \ldots, a_{98}$, which are 3, 4, 5, …, 35.
The third subsequence contains $a_3, a_6, a_9, \ldots, a_{99}$, which are 5, 6, 7, …, 37.

That is, each subsequence is formed by selecting every third term from the original sequence, and contains a progression of consecutive integers. (Note that the first subsequence has one term more than the other two.)

Summing the three subsequences separately, then adding the results, we have

$$\sum_{i=1}^{34} i+\sum_{i=3}^{35} i+\sum_{i=5}^{37} i=\sum_{i=1}^{34} i+\left(\sum_{i=1}^{35} i-\sum_{i=1}^{i=2} i\right)+\left(\sum_{i=5}^{37} i-\sum_{i=1}^{4} i\right)$$

$$=\frac{34\times 35}{2}+\left(\frac{35\times 36}{2}-\frac{2\times 3}{2}\right)+\left(\frac{37\times 38}{2}-\frac{4\times 5}{2}\right)=595+627+693=\mathbf{1915}$$

2. Solution 1: $A^2-AB-BA+B^2=(A-B)^2$. (Note that this does not equal $A^2-2AB+B^2$, as matrix multiplication does not commute.) Then

$$(A-B)^2=\left(\begin{bmatrix}7&2&-8\\3&5&1\\-6&0&-4\end{bmatrix}-\begin{bmatrix}8&1&-9\\2&6&0\\-7&-1&-3\end{bmatrix}\right)^2=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}^2$$

$$=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}=\begin{bmatrix}\mathbf{3}&\mathbf{-1}&\mathbf{-1}\\\mathbf{-1}&\mathbf{3}&\mathbf{-1}\\\mathbf{-1}&\mathbf{-1}&\mathbf{3}\end{bmatrix}$$

Solution 2: Substituting in A and B, we calculate:

$$\begin{bmatrix}7&2&-8\\3&5&1\\-6&0&-4\end{bmatrix}\begin{bmatrix}7&2&-8\\3&5&1\\-6&0&-4\end{bmatrix}-\begin{bmatrix}7&2&-8\\3&5&1\\-6&0&-4\end{bmatrix}\begin{bmatrix}8&1&-9\\2&6&0\\-7&-1&-3\end{bmatrix}$$

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$$-\begin{bmatrix}7 & 2 & -8\\ 3 & 5 & 1\\ -6 & 0 & -4\end{bmatrix}\begin{bmatrix}8 & 1 & -9\\ 2 & 6 & 0\\ -7 & -1 & -3\end{bmatrix}+\begin{bmatrix}8 & 1 & -9\\ 2 & 6 & 0\\ -7 & -1 & -3\end{bmatrix}\begin{bmatrix}8 & 1 & -9\\ 2 & 6 & 0\\ -7 & -1 & -3\end{bmatrix}$$

$$=\begin{bmatrix}103 & 24 & -22\\ 30 & 31 & -23\\ -18 & -12 & 64\end{bmatrix}-\begin{bmatrix}116 & 27 & -39\\ 27 & 32 & -30\\ -20 & -2 & 66\end{bmatrix}-\begin{bmatrix}113 & 21 & -27\\ 32 & 34 & -10\\ -34 & -19 & 67\end{bmatrix}+\begin{bmatrix}129 & 23 & -45\\ 28 & 38 & -18\\ 37 & -10 & 72\end{bmatrix}$$

$$=\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}$$

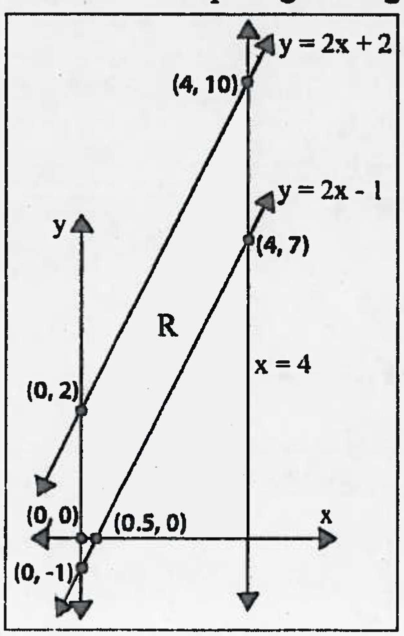
<u>Solution 3</u>: A calculator can be used.

3. Any two regular heptagons are similar; therefore their area will vary as the square of their side lengths. If the area of a regular heptagon with side length 1 is A, then the area of the first heptagon is $7^2A = 49A$ and the area of the second heptagon is $24^2A = 576A$. So the area of the third heptagon is $49A + 576A = 625A = 25^2A$, meaning its side length is $\mathbf{25}$.

4. First we must find the equation of the line parallel to $y = 2x - 1$ that contains the point $(2, 6)$. Since parallel lines have the same slope, the line has slope 2. We can use point-slope form to find the equation:

$$y - 6 = 2(x - 2) \Rightarrow y = 2x - 4 + 6 \Rightarrow y = 2x + 2$$

This line and the given lines can be graphed, as below. We see the figure bounded by all five lines is the pentagonal region marked R. Its area can be found by finding the area of the parallelogram whose vertices are $(4, 7)$, $(4, 10)$, $(0, 2)$, and $(0, -1)$, and subtracting the area of the triangle whose vertices are $(0, 0)$, $(0.5, 0)$, and $(0, -1)$.



The parallelogram has height 3 and width 4, so its area is $3\times4 = 12$, and the triangle has height 1 and width $\frac{1}{2}$, so its area is $\frac{1}{2}\times1\times\frac{1}{2} = \frac{1}{4}$.

Then the area of R is $12 - \frac{1}{4} = \mathbf{\frac{47}{4}}$ sq. units.

5. <u>Solution 1</u>: We can separate the given series into two series, where the first contains all the positive terms and the second contains all the negative terms.

The first series is $\frac{1}{3} + \frac{1}{45} + \frac{1}{675} + \cdots$, which is a geometric series whose first term is $\frac{1}{3}$ and whose common ratio is $\frac{1}{15}$. Then its sum is:

$$\frac{\frac{1}{3}}{1-\frac{1}{15}}=\frac{\frac{1}{3}}{\frac{14}{15}}=\frac{15}{42}=\frac{5}{14}$$

The second series is $-\frac{1}{15}-\frac{1}{225}-\frac{1}{3375}-\cdots$ which is a geometric series whose first term is $-\frac{1}{15}$ and whose common ratio is $\frac{1}{15}$, so its sum is:

$$\frac{-\frac{1}{15}}{1-\frac{1}{15}}=\frac{-\frac{1}{15}}{\frac{14}{15}}=\frac{-1}{14}$$

So the sum of the original series is $\frac{5}{14}-\frac{1}{14}=\frac{4}{14}=\frac{2}{7}$

Solution 2: Note that each pair of consecutive terms (where the first term is positive) is of the form $\frac{1}{3\times 15^n}-\frac{1}{15^{n+1}}$, which is equal to $\frac{5}{15^{n+1}}-\frac{1}{15^{n+1}}=\frac{4}{15^{n+1}}$.

Then the given series can be written as $\frac{4}{15}+\frac{4}{225}+\frac{4}{3375}+\cdots+\frac{4}{15^n}+\cdots$

The sum of this series is $\frac{\frac{4}{15}}{1-\frac{1}{15}}=\frac{\frac{4}{15}}{\frac{14}{15}}=\frac{4}{14}=\frac{2}{7}$

6. Solution 1: We can take the base 7 logarithm of both sides:

$$\log_7 7^{2x+4}=\log_7\left(\frac{1}{49}\right)^{9-2x}$$
$$\Rightarrow (2x+4)\log_7 7=(9-2x)\log_7\left(\frac{1}{49}\right)$$
$$\Rightarrow (2x+4)\times 1=(9-2x)\times(-2)$$
$$\Rightarrow\quad 2x+4=4x-18\quad\Rightarrow\quad 22=2x\quad\Rightarrow\quad x=11$$

Solution 2: We can first rewrite $\frac{1}{49}$ as 7^{-2}:

$$7^{2x+4}=(7^{-2})^{9-2x}$$
$$7^{2x+4}=7^{4x-18}$$
$$\Rightarrow 2x+4=4x-18\ \Rightarrow\ 2x=22\ \Rightarrow\ x=11$$

7. We use the trigonometric identity (specifically, the sine double-angle formula) $\sin 2x=2\sin x\cos x$, to replace the input $\frac{\pi}{12}$ with $\frac{\pi}{6}$, whose sine and cosine is better known:

$$12\left(\sin\frac{\pi}{12}\right)\cdot\left(\cos\frac{\pi}{12}\right)=6\times 2\left(\sin\frac{\pi}{12}\right)\cdot\left(\cos\frac{\pi}{12}\right)=6\times\sin\frac{2\pi}{12}$$
$$=6\times\sin\frac{\pi}{6}=6\times\frac{1}{2}=3$$

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8. Solution 1: First, note that 0 cannot be one of the numbers' digits, since if it were in the hundreds place the number would not have three digits, and if it were in the tens or ones place, the digits can't be in increasing order.

Secondly, at most one of each of the remaining digits (1 through 9) can appear in each number, or the digits would not be strictly increasing.

Finally, note that any group of three different numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ can be arranged in increasing order in exactly one way. That is, the number we are asked for is equal to the number of ways of selecting three distinct digits from the aforementioned set. And this amount is:

$$ {}_9C_3 = \frac{9!}{6!\,3!} = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = \mathbf{84} $$

Solution 2: We can systematically count the numbers. First note that the hundreds digit cannot be 8 or 9. When the hundreds digit (hereafter called A) is 7 and the tens digit (B) is 8, the ones digit (C) can take only one value, 9.

When $A = 6$ and $B = 7$, there are 2 possibilities for C.
When $A = 6$ and $B = 8$, there is 1 possibility for C.
So altogether there are $2 + 1 = 3$ possible three-digit numbers when $A = 6$.

When $A = 5$ and $B = 6$, C has 3 possibilities.
When $A = 5$ and $B = 7$, C has 2 possibilities.
When $A = 5$ and $B = 8$, C has 1 possibility.
So altogether there are $3 + 2 + 1 = 6$ possible three-digit numbers when $A = 5$.

We see the emerging pattern: each next lower value of A results in the next triangular number of total possibilities. (The triangular numbers are those of the form $1 + 2 + \cdots + n$ for some positive integer n.) Then $A = 4$ will result in 10 numbers, $A = 3$ will give 15 numbers, $A = 2$ will give 21, and $A = 1$ will give 28. Summing these,

$$1 + 3 + 6 + 10 + 15 + 21 + 28 = \mathbf{84} \text{ total numbers possible.}$$

9. $105_6 + 342_5 = (1\cdot 36 + 5) + (3\cdot 25 + 4\cdot 5 + 2) = 41 + 97 = 138$

We note that $7^3 = 343$, so we only need to fill the 7^2 place and below.

$$138 \div 49 = 2 \text{ with remainder } 40$$
$$40 \div 7 = 5 \text{ with remainder } 5$$
$$\text{So } 138_{10} = \mathbf{255_7}$$